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Original Article

Continuity Revisited

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ABSTRACT
The present digest is intended to exhibit the historical progression of the notion of 'continuity' from the standpoint of four different signature lines, namely Philosophers, Geometers, Arithmetizers and Topologists.
Keywords: Continuity, axiom, closeness, topology, continua, infinitesimal, continuum.

INTRODUCTION

Substratum of continuity

In calculus the first incident of tossing the word '***continuity***' has been traced with reference to a function $\psi: \mathrm{R} \rightarrow R$. In calculus, we have seen that the limit of a function $\psi(x)$, as $x \rightarrow a$ can often be found by computing the value of the function at the point a. Functions possessing such a property are called continuous at a. The most popular device for continuity, that is prominently being adopted by entire modern mathematical community is mentioned as below.

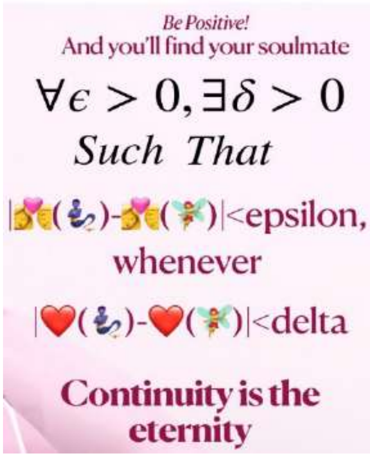

Figure 1: Screen of motivation for continuity

Calculus device for continuity- A function $\psi(x)$ is continuous at a number a if the following three steps hold.

First. $\psi(a)$ is defined (i.e., a lies in the domain of ψ)
Second. $\lim_{x \to a} \psi(x)$ exists (i.e., ψ must be defined on an open interval containing a).
Third. $\lim_{x \to a} \psi(x) = \psi(a)$

However, like the convergence, function's continuity has been a subtle and extremely important notion which is not only utilized in Calculus, but in almost every branch of mathematics. In fact, continuity is probably the single most important concept in all the mathematical premises. Admitting in mind that '***a function is a way to walk from one set to another***', or speaking topologically, ***'a function is a way to transform one topological space into another'***. When the function is continuous, most of the crucial features that the domain space possesses (e.g., like being all in one piece, being open, being closed, being compact etc.) are maintained into their existential form, under the transformation, so that the image space could also retains these features. This kind of preservation of such crucial features is of the utmost importance in topology. Such an act of preservation of crucial features of mathematical objects by the continuity of mathematical function is being practiced by human being in many senses. For instance, now a days we are giving emphasis upon '*sustainable use of natural resources'*-which straightforwardly means- using natural resources without harming the nature, i.e., being aware of "human act (function) to mother nature (domain) so that the quality (being natural) of codomain (mother earth) remain intact"

The pragmatic and splendid outcome of continuity in almost all the mathematical disciplines is that "*any shape, maintaining its continuity can be elucidated by a single equation*". However, if there are fractures or interruptions in continuities of shapes (e.g., sharp edges and singularities etc.), then more than one equation would be needed to define the fractured parts of the shape under consideration. In spite of continuity

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being a highly insisted device specifically offered by calculus and Topology, from aesthetic point of view, it is sometimes felt necessary to break the seamless-ness of shapes, so that more advanced and beautiful shapes could be evolved (e.g., evolving particular geometric structures by breaking the continuity of flat surface material).

The present digest is focused on exploring the concept- 'continuity', from the various dimensions of history. More explicitly, it has been tried to nest the historically generated thoughts, ideas, axioms, definitions and results regarding continuity. In particular, the standpoints of Philosophers, Geometers, Arithmetizers and Topologists have been referred to weave the fabric of the 'continuity'.

2. 'Continuity' in the time of Aristotle and Euclid

Henri Poincaré in 1905 held that- '*primarily, what properties of 'space' are responsible to call the so-called space a 'mathematical space'*? In response to this question, he evoked three of the properties of any mathematical space, namely:

1. It is continuous
2. It is infinite
3. It is of three dimensions

Perhaps, the emergence of clearer ideas of 'continuity' gradually came into full swing from 17th century onwards, wherein the literal meaning of the world 'continuity' had been assumed to be "seamless, unbroken, uninterrupted or ceaseless". And thus, the mathematical entity, which in modern mathematics is called the '**continuum**' is assumed as an '*unseparated or pause-less or cavity free thing'*. Further, it has been heuristically supposed that most of the physically phenomena such as displacement, velocity, growth of living entity etc. are continuous in nature as they vary with time. Even, many philosophers have evoked that space and time and natural processes occur continuously, for instance, Leibniz made a famous argument that "***nature makes no jump***". The geometrical entities such as lines, planes and solids have also been considered either as aggregations of infinitesimal parts or the accumulation generated by the flow of some entity. However, there are the situations where this argument gets infringed- e.g., the discontinuity occurs in case of electric current.

Besides the above contemplation on 'continuity', if we switch back history, we can find a long lasting and vibrant debate over this issue. The very first emergence of ideas of 'continuity' and 'infinitesimal' in mathematics can be found with Greek atomist philosopher Democritus (450 BC) and then with Eudoxus (350 BC). The doctrines, they followed in delineating 'continua as infinitely divisible entity', is now familiar to us as 'divisionism'. At the prima facie, the approach of '***divisionism***' encapsulates a long chain of logics and is being discussed in the following subsection (2.1).

1.1. Prima facie of 'divisionism'

Observations made by Democritus and Eudoxus have been interpreted by (J. L. Bell, 2005a) as follows:

- The very nature of the 'continuum' or the 'continua' is- being indivisible or unbreakable. But the seamlessness or the unity of continua never implies that its ingredients are not divisible. In fact, the indivisibility of continua means that endless recursive division of it is always possible. That is the entity which can be divided everlastingly. Thus, the basic characteristic of continua is that- such an entity can be decomposed into ever smaller entities, provided the process of decomposition never terminates. One can explicitly think of continuum as;

  'continuum' ≈ an entity consisting of entities which are 'continuum' themselves
- The witness, for example in nature, could be the magnet- as if we crush it and even turn it into powder form, even then each of its corpuscle shall retain the property of magnetism. Further, to feel the potential of continuum, one can consult the fractal analysis (see figure-2)

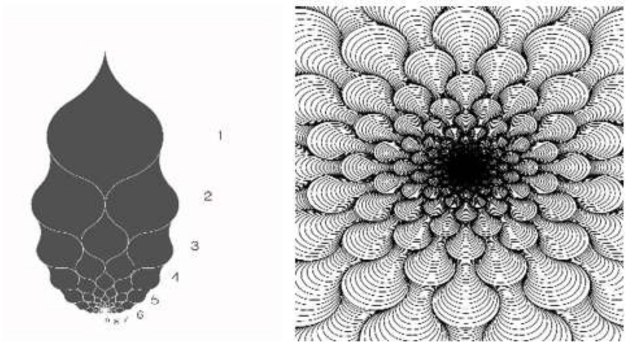

Figure 2 motivational pictures of 'continua' where each part of these pictures can be endlessly divided into ever smaller parts.

- The infinitesimal magnitude of the continuum can be informally conceived as a continuum itself, i.e., the smallest possible parts of the continuum can be though superficially as a continuum.

- The philosophy of continuity gives rise the sense of 'being connected'

2.2 Aristotelian standpoint to continuity

Drawing inspiration from the logical chain of the thought held by Democritus and Eudoxus, Aristotle (384-322 BC) proposed an idea that- 'the theme *infinitesimal* is tangled with the notion of *continuity*'. Soon, this idea led him to the flowing arguments:

- With his geometrical signature line, Aristotle made an observation that- ***"Nothing that is continuous, Aristotle held, can be composed of indivisibles"***. (Evans, 1955)
- Behind this, Aristotle gave the logics that- as time is not constituted of instances, likewise a line is not constituted of points, because both the time and line are continua. Also, by continua, he meant that- ***"which is divisible into divisibles, that are further divisible"***
- He proposed that a continuous magnitude is- '***that allows it to be dissected into infinite number of parts***.
- It was made clear by him that ***though a line cannot be dismantled into infinite number of pieces, even then it is an aggregation of infinite number of points, and thus it retains continuity.***
- From the above supposition, it becomes precise that, as per geometrical viewpoint of Aristotle- the continuity of a line entirely depends upon the continuity of motion.
- He further argued that-continuous magnitude is perceived due to 'motion' and therefore, ***'it is the motion, which is directly responsible for the generation of continuous magnitude'***. For, clear understanding, he mentioned an example- A moving point generates a line and a moving line generates a surface.
- With the above ideas, Simplicius(Urmson, 2014) (Baltussen et al., 2014) rephrased Aristotle's principle in a nutshell as:
  "***A line is the fluxion of the point***"

Eventually, Aristotle in his book 'Aristotle's Physics' ended his quest for continuity with two criteria, as follows:

- **Aristotle's criteria for continuity**: Aristotle said that- something is said to be continuous if
  - ✓ ***The entities or things whose limit, at which they touch, is one***(Lang, 1992)
  - ✓ ***That thing or entity, which is divisible into what is always further divisible***(Sachs, 1995)

3. The Axioms of continuity- in the time of Euclid

The great flux of logics over continuity, continua and infinitesimal, propagated from Democritus (450 BC), Eudoxus (350 BC), Aristotle (384-322 BC) and Simplicius (490-560 AD) led Euclid (300 BC) to refine his fundamental propositions of geometry, which he quoted in his famous book Euclid's Element(Euclid, 1956). Indeed, Euclid found axioms of continuity as a suitable tool to minimize the number of pauses in his postulates of geometry. One of his first postulates, which he refined with the aid of '*principle of circular continuity'* can be taken into consideration as an example here. Consider the following arguments, which Euclid gave to justify his first and foremost proposition:

3.1 First and the foremost proposition of Euclid

Postulate (I): ***"Given any segment, there is an equilateral triangle having the given segment as one of its sides"***(Conover, 2014) (Greenberg, 1993) (Thomas & Thomas, 2003) (Heath, 1926)

Proof: Starting the proof with the fundamental idea on the construction of 'line' would help us understating the more complex proof to the present postulate. (Longo, 2012), (Longo, 2015) has fantastically analyse and synthesize upon the construction of the first fundamental structure of Greek Geometry.

If we simply go through the Euclid's book of Geometry, we can easily observe that in the entire Greek Geometry, the invention of first and fundamental mathematical structure has been the '*line with no thickness.*' In fact, no Euclidean line is possible without acting a trace and without no thickness (Longo, 2015). Clearly a gesture alone or logic alone cannot describe the line. This simply means that 'lines are ideal objects' and thus they can be thought to be *a cohesive continuum with no thickness* (Longo, 2012). In Euclidean geometry- when two thick-less (1-dimensional) lines suitably intersect with each other, produce a point (no-dimensional structure). With these much of fundamental structures, Greek geometry moved towards the invention of continuous lines with no thickness and the geometer called such a construct-an abstract divine construct. (Longo, 2012) synthesized that- no matter, a line is continuous or discrete, it is always a gestalt rather than a set of points.

For the sake of convenience, let us now sketch the proof of Euclid in a step-by-step sequence of logics as follows:

Step 1- Let PQ be any given line segment. Now, with centre P and radius PQ, describe a circle QRS (under Euclid's 3rd postulate, see(Euclid, 1956) (Heath, 1926)) (See Figure 3)

Step 2- Again, by assuming Q as a centre and QP as radius, we can describe another circle PRT using the same Euclid's postulate-III(Euclid, 1956) (Heath, 1926)]. (See Figure 3)

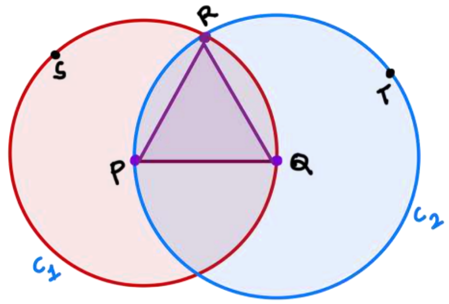

Figure 3: Notion of Circular Continuity under Euclid's postulate-III

Step 3- From a point R, at which the circles C1 and C2 intersects each other, sketch the line segments RP and RQ (under the Euclid's postulate-I(Euclid, 1956) (Heath, 1926)).

Step 4- Now, because P is the centre of circle C1 and Q is the centre of C2, PR will be congruent to PQ (in view of circle's définition).

Step 5- Similarly, Q being the centre of circle C2, clearly QR will be congruent to QP due to the definition of circle.

Step 6- Finally, since RP and RQ are congruent to PQ (due to steps 4 and 5),

Step 7- Consequently, the △ PRQ is an equilateral triangle, having PQ as one of its sides.

Observation- Now if we keenly look back each of the logical steps we outlined above, it seems that the proof is flawless. But observing the 3rd step above, we conclude that our belief on the fact 'that two circles intersect each other at point R 'is due to the diagram drawn (Figure 3). It means if we do not allow ourselves to use diagram, the step 3rd become less explicit and therefore, we need some additional axiom to prove that circles described in the proof of Euclid's first proposition intersect each other.

Thus, to make step 3rd more precise or explicit, let us go through the principle of circular continuity.

Definition-1: Principle of circular continuity

This statement enunciates that ***"If a circle C1 has one point inside and one point outside another circle C2, then the two circles intersect at two points."***

In circular continuity, the notion of '*inside/ outside' a circle* is utilized by stating that a point U is inside a circle having centre O and radius OV if $OU < OV$ and the same point lies outside if $OU > OV$.

The notion of '*inside/ outside*' can be made precise with the assistance of elementary doctrine of continuity, which states the following:

Definition-2: Elementary doctrine of continuity

"Consider a segment of straight line. If one end point of this segment lies inside a circle and other end point outside the circle, then such a segment intersects the circle."

But what makes the above two principles-the continuity principles? The answer must be in the geometry sketch (See Figure A2), wherein a line segment with the help of a pencil is drawn by moving the pencil continuously from a point P to Q. It's very much obvious that such a drawing should traverse a circle having centre say O and it's also natural to say that if, it does not happen like this, there must be a '***hole*** 'present either on the line segment or on the boundary of circle.

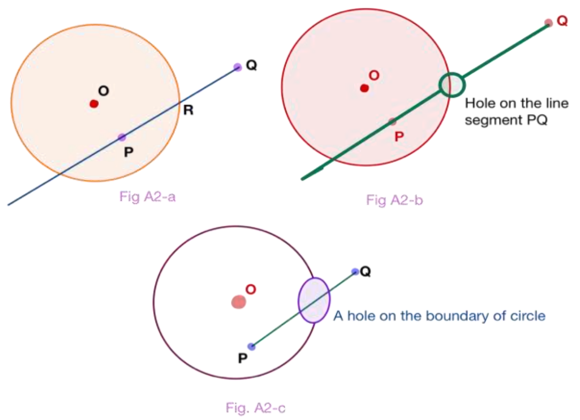

Figure A2: (a): geometrical interpretation of elementary continuity, where a line segment PQ is drawn by moving a pencil and the segment traverses through the circle. (b): geometrical interpretation of discontinuity due to a hole being present on the line segment PQ. (c): geometrical representation of discontinuity due to a hole being present on the boundary of circle.

One strange thing from Euclid's '***Eléments***' that mesmerize the mathematicians is- *the use of actual construction geometry as a device for figures and diagrams bearing certain characteristics.* Thus, geometric constructions were affected by drawing of straight-line segments and circles as per the guidelines of postulates 1 to 3 of Euclid's and the extract of this kind of construction was to determine new line segments, circles and so on from the points of intersections of lines and circles. But, the intersection of such line segments and line segments with circles, so as to determine new lines and circles gave rise a question of ***existence of intersection points*** and thus the quest for a new kind of existential postulate was started.

Killing tried to assist the existence of intersection points by putting forward two rules(Killing, 1892):

3.2 Killing's rules for the existence of intersection point:

Rule-I: Suppose a line belongs completely to a geometric figure which is dissected into two parts; then, if the line has at least one point in common with each part, it must also meet the boundary between the parts.

Rule-II: If a point moves in a figure which is divided into two parts, and if in the beginning of motion, it belongs to one part, and at the end of the motion, it pertains to the other part of figure, then meanwhile the motion of point, the point must reach at the boundary between the two parts.

4. Continuum, infinitesimal and continuity in 17th & 18th centuries

Trailing the ideology of Democritus (450 BC), Eudoxus (350 BC), Aristotle (384-322 BC), Simplicius (490-560 AD) and then Euclid (300 BC), 17th and 18th century philosophers and mathematicians such as, Kepler, Galileo, Newton, Marquis De Hôpital, Leibnitz, Euler, Barrow and Kant tried to establish a systematical construct of relations among continuity, continuum and infinitesimal.

Moreover, especially in 17th century, mathematicians coined the following prominent thoughts:

17th century mathematicians held that-

- *Continuous curves are made up of infinitesimal straight lines' and therefore the continua's (which is thought as a unity) constituent parts must be continua themselves.'*
- *Since entities like points are non-decomposable or indivisible, thus, they cannot be the parts of any continuum.*
- *Any number may be supposed as an infinitesimal number, if it does not coincide with the number zero and if in some sense, it remains smaller than any finite number.*
- *In Newtonian calculus, infinitesimal quantities were treated as 'instrumental'.*
- *From Leibnitz's perspective, the infinitesimal quantities were supposed to be 'unassignable quantities.*
- As per the treatise of Marquis De Hôpital, entitled 'Differential Calculus' published in 1696, it was postulated that- "*a curved line may be regarded as a composition of infinitely tiny straight lines" and "one can take any two quantities equal, provided they are differ by an infinitely small quantity*"
- Isaac Barrow (C.1630-1677), an English mathematician, while developing method for finding tangents realized that- *notion of infinitesimal is an essential tool for his method and thus he introduced two mesmerizing words "**Linelets**" and "**Timelets**" for infinitesimal*, which appeared later in his work "*Lectiones Geometricae*" in 1670.

However, among the 17th century philosophers and mathematicians, the British mathematician sir Isaac Barrow (1630-1677) has been credited as a pioneer in defining the continuous magnitude in a systematic way. Barrow begun to establish a reciprocal relation between the problem of quadrature and that of finding tangents to the curves and he drew the following conclusion:

4.1 Barrow's conclusion(s)

Barrow, in his work "*Lectiones Geometricae*" in 1670 observed that(J. Bell, 2004)-

a. If for any curve $y = \psi(x)$, the quadrature be known with the area given by $\varphi(x)$, then the subtangent of the curve $y = \varphi(x)$ can be determined by measuring the ratio of its ordinate to the ordinate of original curve $y = \psi(x)$, i.e., subtangent of $\varphi(x) = \frac{\text{ordinate of } \varphi(x)}{\text{ordinate of } \psi(x)}$.

b. Because, continuous magnitudes are generated due to motion, therefore they essentially be dependent on time.

Sir Isaac Newton (1642-1727) during the plague pandemic, deeply deployed the tools (a) and (b) of his teacher Barrow, and consequently established his work, now popularized as "Calculus of fluxions". He, thus pave the way to a new paradigm of continuity(J. L. Bell, 2005b).

4.2 Isaac Newton's calculus of fluxions

Here are the notions, that Newton apprehended in his work-

1st. *As Newton supposed the infinitesimal quantities to be just instrumental in his calculus, he notified these instrumentals as "**Momentary increments**". Probably, such notification was because of utilizing kinetic notions in his work.*

2nd. *For him the "momentary increment" means an instance of time, or a moment of time- of abscissa or the area of the curve with the abscissa (abscissa itself stands for time in this case)*

3rd. *Newton introduced the symbols-*

$o \stackrel{\text{def}}{=}$ abscissa, $v \stackrel{\text{def}}{=}$ ordinate, $ov \stackrel{\text{def}}{=}$ area of the curve.

Probably these symbols infer that, Newton's supposed a curve to be a plot or graph between velocity and time.

4th. *Taking into account a moving line or an ordinate, as a moment of area of the curve, Newton established a generalized result for the reciprocal relation between the differentiation and the integration.*

5th. *Finally, in his work "Methodus fluxionum", Newton had introduced the variable quantities generated due to motion as a "fluent". He evoked the rate at which quantities "fluent" were generated, as a "fluxion".*

6th. *The notations, he notified in his calculus were denoted by-*

fluxion of the fluent $\stackrel{\text{def}}{=} \dot{x}$, moment of fluxion $\stackrel{\text{def}}{=} \dot{x}o$

In the meanwhile, Gottfried Wilhelm Leibnitz (1646-1716), a German polymath was intensively working on 'a general law of continuity' and he was literally provoked by the question that- *what composes the continuum?* He cited this problem as "*Labyrinth of the continuum*".

4.3 Leibnitz's monadism and law of continuity(J. L. Bell, 2005a)

G.W. Leibnitz, in his quest for the principle of continuity, walked through the problem that- *whether a continuum can be built from indivisible entities? If yes! Then how*? In search of answer, he put forwarded a new philosophy called "***Monadism***", which led him to the following conclusions-

1st. Leibnitz inquired himself that-*if every real entity is supposed to be either as a 'unity' or as a 'multiplicity', and further, if 'multiplicity' be essentially treated as an accumulation of 'unities.' Then under what characteristics and in what category, a geometric continuum (e.g., a line, surface or a solid) should be placed?*

2nd. He further argued for example, that-*if a line is extended, and an extension being categorized as recursion, then a line cannot be treated as a true 'unity', as it is divisible into parts. Hence, rather a true 'unity', a line is true 'multiplicity'.*

3rd. Leibnitz concluded the 2nd argument under the influence of the already established logics that state- *(a). The only unities of any geometric continuum could be points, but points have no further divisibility attitude and therefore, points are not more than the extremities of the extension of line*

(b). Also, according to Aristotle- "no continua can be composed of points"

4th. With the aid of argument 2nd and 3rd, Leibnitz came to assert that- *"Continua is neither a unity nor a multiplicity", which literally means that, in practice, there does not exist anything like real continuum, which is either unity or an accumulation of unities.*

5th. Eventually, he founded that-*space and time, in ideal situation can though to be as continuum. However, anything which is real (for example matter) always reveals the discreteness as being composed of substance like units and he notifies such simple units as "**Monads**"*

But what makes Leibnitz enforced to think upon infinite numbers? In fact, the answer is hidden in the work of Galileo Galilei, who in his '***Two new sciences***' proposed that (Arthur, 2015)-

a. "The matter is composed of an actually infinite number of atoms."

b. *"And each of the atoms are separated by infinitely small voids."*

Leibnitz followed the idea of Galileo Galilei that 'in the infinite, there is neither greater, nor smaller' and demonstrated this as follows(Leibniz, 2001):

"*Among numbers, there are infinite roots, infinite squares, infinite cubes. Furthermore, there are as many roots as numbers. And there are as many squares as roots. Therefore, there are as many squares as numbers.*"

Leibnitz, from the above quoted demonstration concluded that- there are as many square numbers as there are numbers in the universe, which is impossible. Consequently, in the infinite, the whole is greater than the part, which is the affirmation of Galileo.

To validate Galileo assertion that 'the whole is greater than the part', Leibnitz finally produced a purely mathematical version (Arthur, 2015). What he did in this mathematical version is:

a. He drew a symmetrical diagram (see Figure 4a), which he called Leibnitz's hyperbola with centre A and vertex B. Then he sat AC = BC = a or 1 without loss of generality.
b. Then he tried to determine the area under this symmetrical curve between the line CB & X-axis
c. To do so, he used DE = 1/ AD = 1/ (1-y), which he expanded in terms of power series as: $DE = \frac{1}{AD} = \frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$
d. Then he calculated the area in question by applying the variable line DE to the line AC = 1, which yield: Area (ACBEM) $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. As per the modern mathematics, Leibnitz has integrated the power series as $\int_0^1 (1 + y + y^2 + \cdots)dy$
e. In a similar fashion, he calculated $\text{Area}(CFGLB) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
f. Finally, he subtracted the finite Area (CFGLB) from the infinite Area (ACBEM) to get- $\text{Area (ACBEM)} - \text{Area (CFGLB)} = \text{Area (ACBEM)}$
g. It was more than enough to demonstrate that subtracting an area (i.e., Area CFGBL), which is definite and explicitly perceivable, from the area under the hyperbola (i.e., Area ACBEM) leaves the Area ACBEM intact. Which shows that the whole is greater than the part.

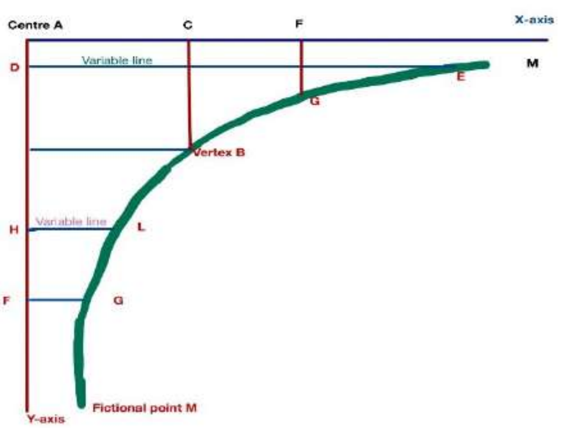

Figure 4a: Leibnitz's Hyperbola

With this philosophy of "monads", Leibnitz presented one of his best doctrines, now known as 'principle of continuity'.

4.4 Leibnitz's principle of continuity(Heath, 1926)

The mathematician contemporary to Leibnitz established some important facts pertinent to the tangent to some given curves, and they granted that 'one could find a tangent line at every point of a curve under consideration'. Basically, they developed a geometric construction wherein they assumed that- given a curve and a point P on the curve, the tangent line can be constructed by passing a line through P and another point Q lying on the curve. Further, any point Q which is differ from P will also yield a line, because in accordance with the Euclid's postulate-I, '*Any two points determine a line*'. To obtain a tangent line, mathematician followed idea of nearness/ closeness as prescribed by Leibniz and they moved point Q close enough/ near enough to P. Such a beautiful construction was later formulated as a general principle and now known as Leibniz's continuity principle:

Definition-3: Continuity principle- *" In any supposed transition, ending in a terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included".*

However, this principle got stuck when few counterexamples came into existence in the course of study. One of them was as follows:

Example 1: Let us consider the function $f(x) = |x|$, where the symbol $|.|$ stands for absolute value which is usually defined as: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. It's obvious from the graphical representation of this function that the function lies in the 1st and 2nd quadrant of the plane. Clearly, for each non-negative value of x, the absolute value function has a tangent that coincides with $f(x) = x$. However, for each negative value of x, the absolute value function has a tangent that coincides with $f(x) = -x$. Now in view of Leibnitz's continuity principle, in the process of drawing tangent, it might be possible to extend towards the origin and hence the origin should be the terminus. However, if we proceed drawing tangent from the right of the origin, the tangent at origin must coincide to the line $y = x$, i.e., the gradient must be +1. Likewise, if we proceed to draw tangent from the left of the origin, the tangent at origin must coincide with the line $y = -x$, i.e., the gradient must be -1. Therefore, the tangent at origin is impossible to define, because a tangent at some given point cannot have two different gradients at the same time.

Apart from the above graphical approach, the same conclusion that '***Leibnitz's continuity principle***' ceased for the function $f(x) = |x|$, can be drawn using the simple device of Calculus as follows:

Example-2: Consider the function $f(x) = |x|$ and let us try to show that Leibnitz's continuity principle fails to hold, specially at (0, 0). For this, let us determine the derivative of given function via implicit differentiation. Suppose $\frac{d}{dx}(f(x)) = \frac{d}{dx}(|x|)$. Making use of chain rule; $\frac{d}{dx}(f(x)) = \frac{df(u)}{du}\frac{du}{dx}$. Where $u = x$, and $\frac{d}{du}f(u) = f'(u)$. Then $\left(\frac{d}{dx}(x)\right)f'(x) = \frac{d}{dx}(|x|)$. Since the derivative of x is 1, therefore $f'(x) = \frac{d}{dx}(|x|)$. Again, in implementing chain rule: $\frac{d}{dx}(|x|) = \frac{d|u|}{du}.\frac{du}{dx}$, where $u = x$ & $\frac{d|u|}{dx} = \frac{u}{|u|}$. Thus, $f'(x) = \frac{x((d/dx)(x))}{|x|}$, and again derivative of x , with respect to x becomes 1, so we finally have: $f'(x) = \frac{x}{|x|}$, which yields +1, when $x \geq 0$ and -1 when $x < 0$

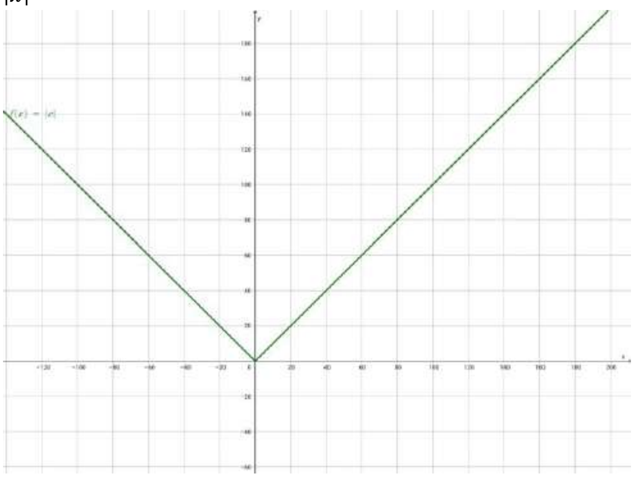

Figure (4b): The gradient to the curve in the first quadrant of the plane is +1, whereas that of the curve in second quadrant is -1. This holds true to each point of either of the curves. In view of Leibnitz's principle of continuity, it is observed that derivative at origin does not exist.

A plenty of such curves having corners or cusps, were soon encountered by mathematicians and they believed that points where derivative did not exist were exceptional. Mathematicians also found hard to imagine the curves which entirely consist of cusps or sharp corners. To imagine and draw jagged curves, prior to calculus it would be heavily required to introduce Topological notions.

Let us switch back to the time, when Leibnitz exploited his continuity principle in developing his infinitesimal calculus. Leibnitz wrote essay "Nova Methodus" in 1684 which was followed by another essay "De Geometri Recondita" in 1686, wherein he formally introduced the notion of differential and integral calculi. In these two essays, he included the following:

I. He supposed a curve characterized by two correlated variables, namely x & y

II. Then, he notified symbols dx & dy to formulate the infinitesimal differences or the differentials between variables x & y.

III. Further, he notified a symbol $\frac{dx}{dy}$ for the ratio of the above two differences and he called this as '*slope of the curve at a point.*'

IV. Though, infinitesimal differences for Leibnitz were unassignable quantities, he in 1684 proposed the following rules without demonstration:
- $d\alpha = 0$
- $d(\alpha x) = \alpha dx$
- $d(x + y - z) = dx + dy - dz$
- $d(xy) = xdy + ydx$
- $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$
- $d(x^n) = px^{p-1}dx$

V. When, Leibnitz felt the essential admittance of incomparably tiny quantities, which were seemingly small than the ordinary numbers- *he argued that, law of continuity governs the incomparably small quantities in the similar fashion as it governs the ordinary numbers.*

VI. Ultimately the argument (V), forced him to involve the infinitesimal quantities with finite quantities. But he always treated the infinitesimal quantities as if they were zero. *Thus, for instance, Leibnitz treated the quantity* $x + dx$ *same as* x. *Such a treatment, he followed was based on the notion that 'differentials cannot be stagnant, rather they should be variables and should be diminishing continuously until arrived at zero'.*

VII. He developed his calculus with the notions that-

- *Differentials or infinitesimal are neither something nor equal to the absolute zero.*
- $dx \approx 0$, *by which he meant that differentials are indistinguishable from absolute zero*
- *Neither* $dx = 0$, *nor* $dx \neq 0$
- $dx^2 = 0$
- $dx \to 0$, *by which he meant that the differential is vanishingly small.*

4.5 Euler's refutation to Leibnitz's monadism

A Swiss polymath Leonhard Euler (1707-1783), while practicing calculus straight away refuted the Leibnitz's monadism by involving the Cartesian principle in his study.

As its well known that- the theory of monads or simple things of which the body is composed, relies upon two general features of the bodies, namely; '*extent & the moving forces*' (Haude & Spener, 1746). Euler argued that, such a theory can be true if the arguments leading to it are valid. With this quest, Euler has set up deepen exploration towards the Leibnitz's theory to arrive at the following concluding remarks:

a. Regarding the first property of monads, i.e., 'extent', Euler said that- it is undoubtedly true that all bodies are composed of parts and that these ever-smaller parts can be distinguished. Because, if through decomposition, we eventually reach at a particle so small that with naked eyes the particle is observed to have no further parts; then it can further be examined with a magnifying glass to discover that the particle still has a large number of real parts.
b. Euler questioned that- whether this long-lasting divisibility can be continued infinitely far? Or whether this decomposition process reaches at some limit, such that there remain particles with no size? Indeed, these questions are still a matter of debate.
c. Regarding the second property of monads, i.e., 'moving forces', Euler proclaimed that- since every-body have such a force to remain in its natural state, the cause of this force must be found in the essence of the body. Thus, it can be rightly concluded that every-body is endowed with a force to remain in its present state.
d. Euler submitted that- the force which is responsible to retain the body in its current state is called '***Vis inertiae'.***
e. He further claimed that in the theory of movement, the force '*Vis inertiae'* is generally a property of the body without which a body ceased to be a body.
f. Thus, Euler gave the reason '**inertia**' for the change in the world, while dealing with the query that '*Why there are continuous changes in the world?*'
g. Euler further refuted the Leibnitz's notion of infinite divisibility by mentioning that- '*Mr. Leibnitz appears to admit infinite divisibility by maintaining that infinitely many monads shall be required to represent the smallest body*'. However, this statement contradicts itself, as this one is equivalent to saying that- 'bodies can through no division, however far this might be continued, be subdivided into such simple things, through which in fact the existence of simple things is denied.'
h. Euler also argued that- if we assume that a body is composed of simple things, then we should acknowledge that the number of these simple things is definite. But if one takes this number to be infinite, it can be no more definite and therefore infinitely huge would mean a magnitude beyond understanding.
i. Thus, Euler summed up the crucial difference between his own and Leibnitz's definitions of infinitely small quantities in the following way (Knobloch,2008):
   (a). Suppose i be an infinitely small quantity, g_q be a given quantity and a_q be an assignable quantity, then:
   (b). Leibnitz- for all $g_q > 0$, there is an $i(g_q) > 0$, so that $i(g_q) < g_q \Rightarrow i(g_q)$ is a variable quantity.
   (c). Euler- for all i and for all $a_q > 0 : i < a_q, \Rightarrow i = 0$

Apart from the above, while refuting the Leibnitz's and Wolff's theory of monads or simple things (Watkins, 2006), Euler has made the following observations against the Leibnitz's monadism(Berkeley, n.d.) (Berkeley, 1754):

1st. He held the Cartesian doctrine – "*the universe if filled with a continuous ethereal fluid*"

2nd. *Universe follows the wave theory of light rather that corpuscular theory proposed by Newton*.

3rd. *The 'infinitesimal' of Leibnitz was refuted by the logic that- any quantity which is less than the magnitude assignable to it, cannot be equal to zero.*

4th. *Euler asserted that the differentials must be zero and thus the quotient* $\frac{dy}{dx} = \frac{0}{0}$, *since for any number* $\alpha, \alpha \times 0 = 0$. *Further, Euler argued that the quotient* $\frac{0}{0}$ *should stand for some number*(Bishop, 1967).

5th. *For natural phenomenon, Euler justified that- a 'minute' like 'infinitesimal' should be considered, provided this minute like quantity must be a concrete element of the continuum.*

6th. *The minute like element of the continuum should not be treated as an atom or monad*

7th. *Finally, the minute like element of the continuum should essentially be divisible.*

4.6 Kantian philosophy of continuity

Unlike the mathematical continuity, Immanual Kant (a prominent 18th century German Philosopher) developed the concept of continuity which was broadly philosophical and was meant to address the nature of our everyday experience in space-time. Because Kant's continuity is keenly associated with his philosophy of space and time and is applicable to the structure of human perception and because the modern sciences like topology, quantum theories etc. are also dealing with the structural aspect of space-time, it is worth quoting Kant to excavate some similarities among various notions of continuity.

Immanuel Kant (1724-1804), in his philosophical stream 'transcendental idealism', gave place to the idea of continuity. As per his ideology, the following logics came into play:

First. There are two aspects of reality- one is "phenomenal realm" and the other is "noumenal realm".

Second. By "phenomenal realm" he meant-things those consist of appearance.

Third. The phenomenal realm is also known as the realm of appearance refer to the world as we experience it through our senses & mental faculties.

Fourth. This realm includes everything we perceive, observe & comprehend through our empirical experience.

Fifth. Thus, as per the Kant's critique of pure reasoning, phenomenal realm is shaped & structured by our cognitive faculties.

Sixth. He, for example argued that- *though 'the space and time are shaped and structured by our cognitive faculties' but they are not the inherent properties of the external world but rather subjective forms that shape our sensory experience and make coherent perception possible*.

Seventh. Kant believed that there is a framework which he called 'spatiotemporal extension', through which humans perceive & understand reality in a perpetual manner.

Eighth. Thus, the things like magnitude which is identified by its appearance can be spatiotemporally extended to infinity and so the spatiotemporal extension of things is continuous (viz. space and time).

Ninth. However, in the phenomenal realm, our knowledge remains limited as we can understand & reason about the objects & events as they appear in space-time, but cannot know things independent of our cognitive processes.

Tenth. By "noumenal realm"- he meant things consisting of understanding to which no objective experience can ever be receive.

Eleventh. He calls this as "things in themselves".

Twelfth. He assumed this realm to be a hypothetical realm of existence that lies beyond human perception & understanding.

Thirteenth. This realm represents things as they are independent of our cognitive faculties or any empirical experience.

Fourteenth. Kant speculated that the noumenal realm contains the ultimate reality of objects and events and thus examples of his realm are often elusive, as we cannot directly perceive them.

If we look at the Kantian transcendental dialectic, we can find the following subtle assertions that Kant made in favour of his continuity principles (Connell, 2022):

1st. Kant has coined the terms **'Quantum Continuum'** & **'Quantum Discretum'**

2nd. Quantum continuum refers to as a whole that is infinitely divisible in virtue of its occupying space.

3rd. Quantum Discretum, on the other hand, is something whose whole has already determined and articulated parts.

4th. Kantian definition of quantum continuum implies that continuity rests on the ability to divide infinitely, without arriving at the smallest entity, while the process of division ensures mereological harmony**.**

5th. Kant in his notion of antithesis has shown that- there are no simple parts in the antithesis. This claim holds, because everything occupies space and since space is not composed of simple parts, everything observed in the space is itself a composite.

6th. In a different way, we from the Kantian 5th argument, can derive that simple parts are not the constituent of space, but the space always has a manifold of elements external to it. That is to say space is not composed of simple parts, but rather **space is made up of space**.

7th. Consequently, everything in space has a mereological structure that mirrors the structure of space and vice-versa.

8th. Kant while dealing with the definition of continuity revealed that one can perform infinite division or decomposition operations, if there is a whole with no simple parts.

9th. Kant further asserted that- the magnitude of continuous space and time can also be viewed as '**flowing**'. This notion of 'flowing magnitude' was also considered by Newton to irradicate difficulties arise due to Leibnitz's monadism.

Besides the above, Kant has proposed the law of continuity of alterations. This law includes two famous continuity principles called law of extensive continuity and law of intensive continuity. Kant has dealt with these laws by following Leibnitz and suggested the following (Jankowiak, 2020):

One. **Law of continuity of alteration-** This law refers to as a metaphysical principle and this states that 'whenever at any instant an object changes its state from one to another, it passes through a continuum of infinitely many intermediate states along the way, i.e., quantum continuum'.

Two. Kant followed Leibnitz's general law of continuity that '*nature makes no leaps*' to deal with his metaphysics and claimed that there are as many particular laws of continuity as there are sorts of leaps that nature refrains to make.

Three. **Law of continuity of extensive magnitude**: Kant proclaimed that- '*Extensive magnitude is a quantity in which a part is able to express the whole*'.

Four. He took examples of space and time as extensive magnitude and asserted that space and time are continuous as no part of either of these extensive magnitudes is a smallest unit.

Five. Kant further argued that- extensive magnitudes as a whole are continuous themselves, because these wholes can be divided into arbitrarily small parts.

Six. **Law of continuity of intensive magnitude:** Kant argued that- an intensive magnitude is a quantity 'which can only be caught as a unity and whose multiplicity can only be represented by approximating it negation equal to zero'. Thus, Kant stated the law of continuity of intensive magnitude as- '*between any two degrees of intensity, there will always be infinitely many more*'. And therefore, Kant defined the intensive magnitudes as continuous.

Seven. Kant also claimed that- 'all realities are intensive magnitudes' and listed fundamental physical properties e.g., weight, volume, temperature etc. as realities, fundamental forces that constitute material and psychological entities such as sensation are all realities.

5. Continuum, infinitesimal & continuity in 19th century

Almost the entire period of 19th century can be admitted as a golden period, during which the base of entire modern calculus was fostered. Seemingly well-grounded logics to define the concepts of continuity, continuum and infinitesimal have been proffered by mathematicians and philosophers of the time. In this section, we limit our focus on fewer, but historically prominent mathematicians and physicist, due to whose work, modern mathematical community became aware of grasping the Hausdorff topological treatment to continuity.

5.1 Means of continuity for Bernhard Bolzano

Historically, Bernhard Bolzano (1781-1848) an Austrian-Hungarian philosopher and mathematician has been crowned as a pioneer of continuous function. Indeed, he is the sole fellow, who posed '*continuity*' in the mathematical world of his time. Seemingly, he was interested in the literal meaning of '*continuity*'. Though the geometric sketch approach used in 'Elementary principle of continuity 'appealed a lot to Leibnitz and Newton in many Calculus discourses, but the '***Pencil-definition'-"A function from D ⊂ R to R" is continuous if we can draw its curve without lifting our pencil from the paper'*** did not always find to be relevant in many 'continuity determination' cases. That is why Bolzano's continuity depended upon the notion of "***closeness***". With this notion, he made the following arguments-

First. In contrast to Pencil-definition, continuity can be determined by taking care of '***closeness***' of function.

Second. Closeness of the function has to be taken in the sense that- tiny modification to the domain of function led tiny changes in co-domain.

Third. In his popular work "Rein analytiĉhe Beweis" in 1817, he introduced the continuity of a real valued function $f(x)$ at some point x as the difference $f(x+t)-f(x)$ which can be treated smaller than a pre-assumed quantity α, provided the quantity t is taken to be small enough.

Fourth. The third argument of Bolzano has been reinterpreted by many mathematicians and it has now been accepted by the community as Bolzano's definition of continuity.

Bolzano's Definition-4: *"If f is continuous, then for each x_0 in the domain $D \subset R$" and for each positive number ϵ, there exists a positive number δ, such that whenever the distance from x to x_0 remains less than δ, the distance from $f(x_0)$ to $f(x)$ will be less than ϵ".*

5.2 Cauchy's perspective of continuity

Augustine Louis Cauchy's (1789-1857) remarkable contribution in introducing the fundamental role of infinitely tiny quantities in the development of infinitesimal calculus has been a mile stone in the history of mathematics. The concept of infinitely tiny quantity was introduced by himself in his text book '***Cours d' analyse-1821***'. Cauchy, in this text book, has asserted the following (Laugwitz, 1987):

One. The infinitely tiny quantities should be denoted by variables whose limit shall assumed to be zero.

Two. These infinitely tiny quantities can be represented by sequences that converge to zero.

Three. These infinitely tiny quantities can also be represented by functions, which vanish at zero

The above three assertion led Cauchy to hypothesize the importance of infinitely small quantities for the treatment of continuous functions. However, Cauchy in 1821 and then in 1833 have introduced two controversial theorems pertinent to continuity, which are as follows:

a. **Theorem 1 (Year 1821)**: *"If a function of several variable is continuous in each one separately, it is a continuous function of all the variables".*
b. **Theorem 2 (Year 1833)**: "*The sum $S(x)$ of a convergent series of continuous functions $U_n(x)$ is itself a continuous function*"
c. Both these theorems seem to be incorrect when interpreted under the conceptual framework of analysis. However, both the theorems become correct when treated under modern theories of infinitesimal.
d. Cauchy summarize the outcome of these theorems as- '***A function is continuous if an infinitesimal change of variable produces an infinitesimal change of the function itself***. However, Cauchy's purely mathematical definition of continuous function is mentioned in the subsequent paragraph.

Cauchy defined continuous function in a more concrete fashion than his predecessor by involving the 'infinitesimal' in a more robust sense. In his famous manuscript "Cours d' analyse", he contended the following(Bolzano, 2012)-

First. Cauchy, in defining the continuity of a function, involved 'infinitesimal' as a variable quantity.

Second. The 'infinitesimal' notified as a variable quantity was treated by him as- 'the quantity whose value decreased indefinitely'

Third. The 'indefinite continuous decrement in the value of variable' must be such that, the variable's value ultimately converge to zero.

Fourth. Finally, with these presumptions, a definition of continuity came into play, which is known as **Cauchy's continuity principle-**

Definition-5: *"The continuity of any function $f(x)$ in some neighbourhood of a value a involves the condition that-* $\lim_{x \to a} f(x) = f(a)$*"*

5.3 Weierstrass approach to continuity

Karl Weierstrass (1815-97) has carried forward the ideology of Bolzano, i.e., 'continuity in terms of closeness' and provided the following robust arguments-

1st. Intuition of continuous motion in the universe can be well understood through arithmetical approach.

2nd. Thus, rather thinking of Infinitesimal as 'variable quantity', we can take it as a 'static quantity'.

3rd. Because, a variable is simply a symbol notified for an arbitrary member of a given set of numbers.

4th. A continuous variable should be like a symbol, whose corresponding set G should have the property that any interval I around any element $g \in G$ contains elements of G, other than g.

5th. With these arguments, Weierstrass succeeded in producing a fantastic definition of continuity, which in modern era known as epsilon-delta criterion of continuity(Boyer, 1940) (Benjamin, 1968). This criterion is being discussed here as definition-6 and is schematically represented as Figure 5.

Weierstrass $\epsilon - \delta$ criterion of continuity

Definition- 6: "*Let $D \subset R$ and let $f: D \to R$. Then f is continuous at a point $x_0 \in D$ if for every positive real number ϵ, there exists a positive real number δ, such that: $|f(x) - f(x_0)| < \epsilon$, whenever $|x - x_0| < \delta$.*"

The given function is then said to be continuous on D, if it is continuous at every point of D.

This definition is *very subtle* as it makes the graph of the function mathematically rigorous. A better way to think of it is- think it in terms of "*closeness*".

A continuous function preserves closeness, in the sense that points which are close to x_0 in the domain of continuous function are sent to the points $f(x_0)$ in the range. The idea of closeness is made precise by asking and answering a question, namely; *how close to x_0* is close enough for points x to be, so that their images $f(x)$ are within a prescribed distance (within ϵ) of $f(x_0)$? The fact that question can be answered (i.e., an ϵ can be shown to exist), no matter how small the prescribed distance ϵ is, means that the function is continuous at the point x_0- it preserves closeness. Here are few examples, mentioned as a witness for Weierstrass approach.

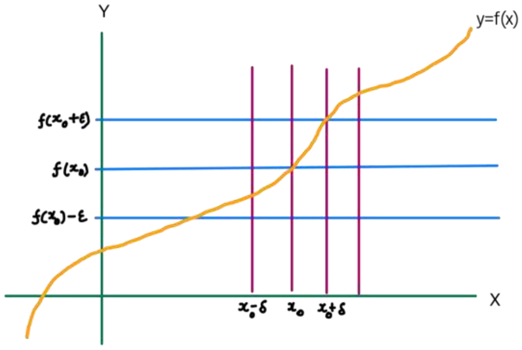

Figure 5: $\epsilon - \delta$ criterion of continuity: If x_0 is any point within the interval of length 2δ with centre x_0 on X-axis, then $f(x_0)$ will lie within the interval of length 2ϵ centred at $f(x_0)$. Thus, if for every value of ϵ, there exists a δ in such a way that this condition holds, then the function $f(x)$ will be called continuous at x_0

Example 3: Consider the function from R to R given by $f(x) = \begin{cases} 1 \; if \;\; x \geq 2 \\ -1 \; if \; x < 2 \end{cases}$. This function is clearly not continuous at $x = 2$

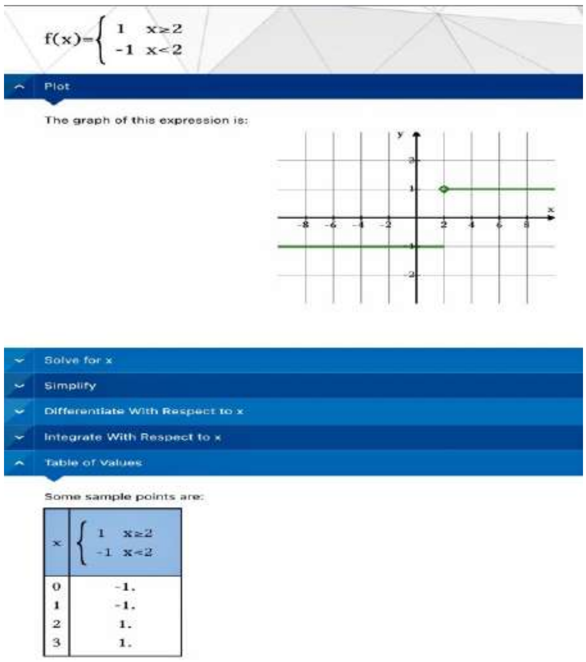

Figure 6-Graph of the function $f(x)$ showing discontinuity at $x = 2$

The Figure 6 exposing that this function is not continuous at $x = 2$, which means, there are points that are close to $x = 2$ that are not sent to points that are close to $f(2) = 1$. Thus, this function is not continuous at $x = 2$, because it fails to preserve '**closeness** 'at $x = 2$.

To pursue the '***closeness*** 'idea a little further, consider the function defined by:

$g(x) = \frac{1}{x}$,if $x > 0$.

This function is continuous on its domain (zero is not in the domain!), its graph looks like (Figure 7):

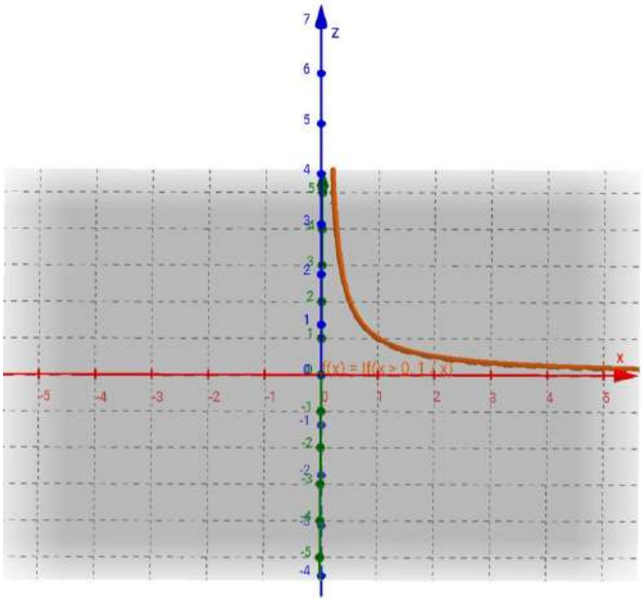

Figure 7: the graph of a function g(x)=1/x, x>0, revealing its continuity at every point of its domain

Since the function given above is continuous, this function must preserve the closeness. But, "close" is a relative term, for example if $x = 1/10$ then $g(x) = 10$; the points in the interval $(1/11,1/9)$ are pretty close to $x = 1/10$, but function *g* spreads these points over the interval $(9,11)$ on y-axis and there are the points in $(9,11)$ almost a whole unit away from $g(1/10)$= 10.

As another example, the points $x = 0.004$ and $x = 0.005$ might said to be very close together (the distance between them is only 0.001 unit) but $g(0.004) = 250$ and $g(0.005) = 200$, so the images under g of these two nearly indistinguishable points are 50 units apart!

Similarly, the points $x = 10^{-6}$ and $x = 10^{-7}$ are so closed together, that you need a powerful microscope to tell them apart. But their images under this continuous function are 9,000,000 units apart!

But this function g is continuous under the provision that $x > 0$.

5.4 Dedekind's postulate for continuity principle

Having inspired from the Weierstrass ideology, Richard Dedekind (1831-1916) has further nourished and formulated continuity of functions by involving reals number system therein. Before going through the precise definition of continuity proposed by Dedekind, let us discuss what conceptions he made to arrive to his continuity definition-

1st. The catalyst for Dedekind's notion of continuity was a query that argues- what is that, which discriminates discrete domains from continuous one?

2nd. In this quest, he found that Weierstrass arithmetical approach could be a great tool to attack on continuity.

3rd. While following arithmetical approach, he involved the set of real numbers in his study.

4th. Involvement of real numbers led him to recognize the 'density property' possessed by the ordered sets of rational numbers.

5th. Further, he recognized that the property of density of rational numbers is inadequate to confirm the continuity of any function or entity.

6th. In his famous manuscript 'continuity & irrational numbers' published in (1872), he mentioned that- it is impossible to associate rational numbers to each point of a straight line. Because there are uncountably many points on the straight line, so rational numbers cannot supply the complete association to the points of straight line.

7th. Thus, it seems that a kind of discontinuity, incompleteness and separation exist in the set of rational numbers. Unlike the set of rational numbers, there exists continuity, completeness and inseparability in straight lines. Dedekind, found that such a situation is beyond proof and therefore can be ascribed as axiom.

With these conceptions, Dedekind has delineated the '***Principle of Continuity*** 'with special reference to his postulate which is as follows:

Dedekind's postulate that assists the consistency of principle of continuity was based on the following popular proposition:

Proposition-1: *If a straight-line segment AB(say) is divided into two parts such that:*

1. Every point of the segment AB belongs to either of the parts
2. The terminus or extremity A pertains to the first part and B belongs to the second part
3. And any point whatever of the 1st part precedes any point whatever of the 2nd part in the order AB of the segment,
There exists a point C of the segment AB (which may belong to either one or to the other), such that every point of AB that precedes C belong to the first part, and every point of AB that follows C belongs to second part in the division originally assumed.

The original postulate that Dedekind proposed have the following form(Euclid, 1956) (Heath, 1926)**:**

Dedekind's Postulate-1: *"If all points of a straight line fall into two classes, such that every point of the first class lies to the left of every point of second class, there exists one and only one point which produces this division of all the points into two classes, and the division of straight line into two parts"*

To elaborate Dedekind's axiom as delineated above, consider the figure-8 and suppose that:

There is a set $\{l\}$ of all points of a line L. Further, let this set be the union of two mutually disjoint non-void sets say G and H, i.e., $\{l\} = G \cup H$, such that no points of either subset are between two points of the other. Then there must exist a unique point say O on the line L, such that one of the subsets is equal to the ray of L with vertex O and other subset is equal to the complement.

Figure 8: A line L={l} which in accordance with Dedekind's axiom is the union of two non-void mutually disjoint sets namely G and H such that G= ray of L with vertex O and H=complement of ray of L with vertex O.

Actually, Dedekind's axiom is the converse of the famous line separation property which states that:

Line Separation Property-1: "*Any point O on L separates all the other points on L into two portions; one is the set contains all the points those lie on the left of O, and other portion is the set contains all those points that lie to the right of O. Also, L is the union of two rays G and H emerging from O*".

However, Dedekind's axiom in contrast to line separation property can be briefly states as:

Dedekind's Postulate-2: "*Conversely, any separation of point on L into left and right portions is produced by a unique point O. A pair of subsets G and H with features as delineated in Dedekind's axiom is called a Dedekind Cut of the line L*"

This postulate can be symbolized as Dedekind's axiom of continuity in the following way:

Dedekind's continuity axiom-1: *Given any Dedekind cut* (Σ_1, Σ_2) *of a line* l, $\exists$ *a unique point P on* l *and an ordering of* l *such that* $\Sigma_1 = (-\infty, P]$ & $\Sigma_2 = (P, \infty)$ *or* $\Sigma_1 = (-\infty, P)$ & $\Sigma_2 = [P, \infty)$.

Here, by Dedekind cut, we mean the cuts as described in Hilbert's axiom-which axiomatize that:

Hilbert Axiom-2: *"Let* l *be a line and let* $\Sigma_1, \Sigma_2 \subseteq l$*, we say that* (Σ_1, Σ_2) *is a Dedekind cut of* l*, if* Σ_1, Σ_2 *are two nonempty convex subsets of* l*, such that-* $l = \Sigma_1 \cup \Sigma_2$ & $\Sigma_1 \cap \Sigma_2 = \phi$ *"*

The motive of Dedekind's axiom was to justify the continuum hypothesis that could ultimately led the avenue of continuity.

Finally, this became true when Dedekind's axiom ensured that a line L is not simply an aggregation of points, but a rather superior structure called '***continuum***'. In addition to this, Dedekind through his axiom revealed that-

- '*A line L has no holes in it*'.
- Of course, without Dedekind's axiom, it would be a tedious situation to any mathematician to think about the existence of π, the exponential constant *e* and the divine ratio ϕ etcetera.
- Apart from this, Dedekind's axiom helped in introducing rectangular coordinate system in the plane and doing geometry analytically.

5.5 Cantor's perspective to continuity

George Cantor focused on Leibnitz's confirmation to the actual infinite, as Cantor found that this confirmation can built a solid base for his theory of transfinite numbers. Actually, Cantor was agitating with the Aristotelian orthodoxy in the philosophy of mathematics. Thereby, a claim of Leibnitz that- '*there are actually infinitely many created substances (monads)*' became a strong support to Cantor to defend his theory of transfinite in contrast to Aristotelian argument that- 'infinity can exist only potentially'. With the aid of Leibnitz's arguments, Cantor designed his theory of transfinite numbers by proposing that-'if there are actually infinitely many creatures, then there must be a corresponding infinite number of them' (Arthur, 2015). Cantor by finally announcing himself a follower of 'Organic Philosophy' of Leibnitz held that-

1st. To satisfactorily describe the nature, one need to theorize the ultimately simple entity of matter

2nd. These ultimately simple entities must be infinite in numbers

3rd. As per the Leibnitz's theory of monadism, these ultimately simple entities shall be called monads or unities.

4th. There can be two different types of matters interacting with each other, which we call **(1) Corporeal matter (2) Aethereal matter**

5th. With these two different classes of matter, one can frame the hypotheses of the powers, namely 'power of corporeal monads and the 'power of aethereal monads'

6th. Corporeal monads as the discrete unities, are equinumerous with the natural numbers and thus have a cardinality or power χ_0, where χ_0 is the first transfinite cardinal number.

7th. The aether, which is supposed to be continuous is composed of aethereal monads is equinumerous to the points on a line and thus equal to χ_1, the second transfinite cardinal number. This second cardinal number has been assumed to be the power of continuum.

It will not be prejudice to say that George Cantor (1845-1918), the German mathematician was the most visionary Arithmetizer among those contemporary to him. A sheer volume of his work on the classification of continuum in terms of infinite sets led him producing the theory of transfinite numbers. Cantor's arithmetization of continuum had the following consequences-

First. An infinite point set is that-which can be put in an 1-1 correspondence with a proper subset of itself.

Second. *Geometrically, a set of points of any pair of straight lines (though one of them is infinite in length) can be placed into one-one correspondence*.

Third. Prior to Cantor's ingenious method of 1-1 correspondence, it was recognized that-infinite set of points have no well-defined size.

Fourth. But, Cantor's approach of taking infinite point sets on a linear continuum equipped with a domain of numbers opened the doors for size comparison of infinite sets in a definite manner.

Fifth. Cantor, then describe an n-dimensional arithmetical space S_n as a set of all n-tuples of real numbers $(r_1, r_2, \dots, r_n)$. Further, he promoted arithmetical point sets in arithmetical space S_n as- *"an accumulation of elements of the elements of space"*

Sixth. In order to enrich the definition of continuum, Cantor took aids of perfect sets and derived or derivative sets and thus held that- "*Continuum is a perfect connected set*".

Seventh. While involving real number system, Cantor defined that- "*The continuum of real numbers cannot be into the 1-1 correspondence with the set of real numbers*" and alternatively he stated that- "infinite sets come in different sizes".

Eighth. With the assistance of first and second arguments of Cantor, he has shown that- there exists a 1-1 map between the point of unit interval and the points of unit cube. He further disclosed that-there exists a 1-1 correspondence between the points of unit interval and the points of unit hypercube (a cube in n-dimensional space such that n>3).

Ninth. He finally wrote a letter to his friend Dedekind and there he said- "***I see it, but I don't believe it***". In response to this, Dedekind wrote- "Don't worry, your correspondence is almost everywhere discontinuous"

5.6 Continuity in view of Boltzmann & Poincaré(van Strien, 2015)

Both Boltzmann and Poincaré admitted that, to adopt the applicability of continuity in Physics, there should be a strong base of empirical approach for the fundamental axioms of Mathematics. They ultimately argued that- in order to determine the status of differential equations in Physics, one needs to have a clearcut justification for mathematical continuity conditions. In search of justification of the applicability of mathematical continuity in Physics, they found that their notion of continuity & discreteness in nature is swinging in between the Kantian philosophy and the Leibnitz's monadism and ultimately, they arrived at contrary to each other. The arguments they posed are as follows:

1st. Poincaré emphasized that- Physicist should rely on the continuous representation of nature and therefore differential equation can be the best tools to aid the study of all the physical phenomena, as during that time it was assumed that the fundamental laws of Physics can be directly interpreted via differential equations.

2nd. Contrary to Poincaré, Boltzmann argued that Physicist must adopt the notion of discreteness of nature.

3rd. But, during 19th century, the differential calculus was robustly developed with a strictly mathematically (without empirical) leveraged continuity principle.

4th. Therefore around 1900, both the Physicists submitted that- the axioms of mathematical continuity are not favourable with the Physically described phenomenon and thus the applicability of differential calculus in physics is problematic until we find a proper empirical base to it.

5th. However, in respect of continuity principles, both the physicists diverged as follows:

Poincaré: the continuity principles are necessary and therefore physicist should work with continuous representation of nature, so that fundamental laws of calculus could be easily applied to the physical phenomenon.

Boltzmann: emphasized that continuity principles are problematic, hence we should go with discrete behaviour of nature, though if required, we can use continuous models as approximations.

6th. The diverged opinions of Poincaré & Boltzmann led them to two distinct continuity notions of differential calculus:

(a). Continuity of possible values: This notion requires a variable that ranges over certain continuous values corresponding to the real numbers.

(b). Continuity of change: This idea requires a relation between the variable which can be represented as a continuous & differentiable function.

7th. Boltzmann did not settle down with the idea of continuity of possible values as he was still relying on discreteness of nature. But, Poincaré agreed upon this notion by arguing that- '*variables in physics can range over continuous values.*' Poincaré further argued that- if we suppose that the physical quantities range over any continuous values, and if these quantities are connected by relations which are usually differentiable function, then we can represent the fundamental laws of physics in terms of differential equations.

8th. Besides the above, Poincaré claimed that determinism in physics can be achieved through the applicability of differential equation. Because, such equations under certain initial conditions produce unique solutions.

9th. Regarding physical continuum, Poincaré claimed that it is fundamentally different from the mathematical continuum, as it is experienced directly. He cited the physical continuum as '***a kind of fusion of neighbouring elements***.'

10th. The fusion of neighbouring elements, which Poincaré called physical continuum, was empirically demonstrated by him as: 'we cannot distinguish tiny differenced weights of bodies through muscular sensation.' That is, it very much impossible to distinguish a 20-gram weighted body from that of 21 grams by lifting them. But we can sense the bodies of weights 20-gram & 22-gram by lifting them. Thus, he claimed that sensation of the weight is non-transitive, and hence muscular sensation never corresponds to numerical values, i.e., if $P = 20\ gram$, $Q = 21\ gram$, $R = 22\ gram$, then muscular sensation of these weights will hold the no-transitivity logic, i.e.,

$$\overset{20\,gm}{\widetilde{P}} = \overset{21\,gm}{\widetilde{Q}}\ \&\ \overset{21\,gm}{Q} = \overset{22\,gm}{\widetilde{R}} \xrightarrow{Muscular\ Sensation} \overset{21\,gm}{\widetilde{P}} < \overset{21\,gm}{\widetilde{R}}$$

11th. Boltzmann & Poincaré's views on continuity of change- As per the assumptions of Boltzmann & Poincaré, a second kind of continuity that emphasizes the applicable of differential calculus in Physics was- "*Change in nature takes place in a continuous manner*."

12th. Under this version of continuity, Boltzmann encountered a problem while working with his entropy function for a thermodynamic system. Actually, he found himself in an awkward situation, when he analyses his H-curve of entropy. He noted that his entropy curve holds the property of non-differentiability and thus bears no physical significance like the Weierstrass continuous but non-differentiable function. However, for justification, Boltzmann argued that 'I do not find any reason of rejecting the entropy function because of its continuous but non-differentiable nature'.

13th. On the other hand, Poincaré's attitude towards the Weierstrass function and Boltzmann entropy function was quite strange. One way, he admired the Weierstrass for his idea of continuous but non-differentiable function. On the other way, he proclaimed that- 'Weierstrass function has irradicated a century long intuition of mathematicians, which is- *continuous functions always have derivatives*.' That is to say, continuous function need not be always differentiable.

14th. Ultimately, Poincaré in 1905, held that- The possibility of science relies on the continuity & differentiability of a function. Thus, to every science, continuity is a priori principle.

6. Topological facets of continuity

Following the idea of '***closeness/ nearness***' ascribed by Bolzano, the 19th and 20th century topologists discovered some new avenues to continuity in topology, so that topology could be evolved in a more sophisticated way. 19th and 20th century topologists admitted that continuity can assist them in handling certain properties of topological spaces. They found, when a topological space under certain continuous function is transformed into another topological space, many of the properties of original topological space remain invariant or preserved. A German mathematician Felix Hausdorff (1868-1942) has been one of the famous fellows, who by admitting the importance of continuity in topology, pioneered the modern form of topology. In the subsequent subsections, we shall include some popular definitions, examples and results on continuity with special reference to topology.

6.1 Felix Hausdorff- A redefiner of Bolzano's continuity

Hausdorff took the definition of continuity as described by Bolzano in a very mesmerizing sense. Instead of involving the notion of closeness/ nearness in terms of distance between objects of a set, Hausdorff has introduced the notion of '***neighbourhood***' and asserted the following:

First. He assumed X & Y to be two topological spaces, which he notified as set of points equipped with a class of neighbourhoods satisfying Hausdorff axioms.

Second. He, further stated in one of his axioms that- the class of all neighbourhoods around all the points of X form a countable class

Third. Finally, with the aid of his topological axioms, he defined the continuity as below:

Fourth. **Definition 7(a)**: Let f be a function with domain X and co-domain Y. Then $f: X \to Y$ will be continuous at any $x_1 \in X$, if for any neighbourhood V containing $f(x_1)$, $\exists$ a neighbourhood U containing x_1, such that $f(x_2) \in V$, whenever $x_2 \in U$.

Fifth. However, Hausdorff's continuity principal can be re-stated as:

Definition 7(b): Given a neighbourhood V of $f(x_1)$, if it is always possible to find a neighbourhood U of x_1, such that f transforms U into V, then f is called continuous at $x_1 \in X$.

6.2 Synchronicity of Hausdorff and Bolzano's ideas of continuity

This subsection is meant to illustrate the equivalence between Hausdorff and Bolzano's definitions regarding the continuity of the function. Indeed, Bolzano and Hausdorff's ideas become in synchronicity, whenever the Hausdorff's notion of 'neighbourhood' is expressed in terms of nearness/ closeness, i.e., distance.

The synchronicity can be achieved by arguing the following:

- Suppose f be a real valued function with some domain of definition (domain to be considered as a subset of set of real numbers).
- Now, in reference to Bolzano's definition (Definition-4), Hausdorff's definition (Definition-7a, 7b) can be delineated by introducing the neighbourhood as follows:
- **Definition 8:** If we are given a neighbourhood, i.e., an open interval of width 2ϵ centred at $f(x_0)$ and we symbolize this as $N(f(x_0), \epsilon)$. Further, if there exists a neighbourhood $M(x_0, \delta)$ of width 2δ centred at x_0 such that- f transforms $M(x_0, \delta)$ into as $N(f(x_0), \epsilon)$, then f is continuous at the element x_0. Likewise,

starting with the Hausdorff's definition, we can arrive at the Bolzano's definition by proceeding bottom to top.

6.3 Topological continuity of 21st century

Besides Felix Hausdorff, there were many other mathematicians, who from various other perspectives tried to enrich the literature of topological continuity. Among those, the German astronomer August Ferdinand Möbius (1790-1868), the German mathematician Adolf Hurwitz (1859-1911), the French mathematician Henri Poincare (1854-1912), the Polish topologist Kazimierz Kuratowski (1896-1980) and the Polish mathematician Waclaw Sierinski (1882-1969) have been immensely recognized for their efforts in tuning the topology.

With their charismatic work, the 21st century topological continuity and topological continua have been unveiled in the more explicit form and thus includes the following definitions and propositions:

Steven G. Krantz in his famous book 'Essentials of topologies with applications evoked that "*The heuristic model for a continuum basically encapsulates the image of a curve in the plane and thus a continuum is a set that, should have no cuts or breaks*" (Krantz, 2009). He defined the continuum with the aid of topology as-

Definition 9: Continua- *If X be a compact connected Hausdorff space, it must be a continuum*(Krantz, 2009). Examples for witness can be; the unit interval $I = [0,1]$, the unit circle in the plane and the torus.

Actually, the definition of continuum (i.e., Definition 9) is simply a generalization of the definition given by(Hocking, 1961) and is delineated as:

Definition 10: Continuum is defined as- a compact connected subset of a topological space.

In modern topology, the prominent device for comparing and contrasting topological spaces are the continuous mappings, provided such mappings are the functions that carry values in a space rather than real or complex numbers. 21st century topologists have put forwarded the notion of continuity by involving the inverse images of such mappings. The most customarily used definition of continuous function includes the following:

- Let $f: A \to B$ be a mapping and let $S \subseteq B$. Then the set $f^{-1}(S) \equiv \{x \in A \mid f(x) \in S\}$ is called the inverse image of the set S under the mapping f.
- **Definition 11a (Modern definition of continuity):** Let (X, τ) and (Y, τ_Y) be two topological spaces. A function or mapping $f: X \to Y$ is called continuous if, whenever $V \subseteq Y$ is open, then $f^{-1}(V) \subseteq X$ is open.

The same can be reinterpreted as:

- **Definition 11b (New Definition of continuity):** Any function $\psi: X \to Y$ is qualified to be called as continuous function if, inverse image of every open set in Y is open in X"

These definition of continuity in topological sense held an important proposition regarding the equivalence between these and Weierstrass traditional definition of continuous function.

Proposition 2: On the real line (with respect to standard topology) the traditional definition of continuity (i.e., Weierstrass Definition 6) is equivalent to the modern definition (i.e., Definition 11a, 11b).

Let us now include some examples based on the equivalence of traditional continuity and modern continuity definitions.

Example 4: Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be given by $f(x) = x^2$. Discuss the continuity of f.

Solution: From heuristic model of continuity in pre-calculus as well as from the traditional model of continuity in calculus, we can easily observe that f is continuous in both sense- i.e., we can draw its graph in a single stroke without lifting pencil from the paper. Also, being a polynomial, f is continuous. However, here we try to examine its continuity in view of modern topological definition.

For, let us take V to be an open subset of the range space $\boldsymbol{R}$. Moreover, we take V to be an open interval $I = (a, b)$. Since, any open set is simply a union of open intervals. Then-

- If $0 < a < b$, then $f^{-1}(I) = (\sqrt{a}, \sqrt{b})$ and that is an open set.
- If $a < 0 < b$, then $f^{-1}(I) = (0, \sqrt{b})$ and again that is open set
- If $a < b < 0$, then $f^{-1}(I) = \phi$ which is again an open set.
- Thus, we have shown that f is continuous with reference to the modern topological definition of continuity.

Example 5: Assume that a set $V \subseteq \boldsymbol{Q}$ is open if there is an open $U \subseteq \boldsymbol{R}$ in the usual topology, so that $U \cap$

$\boldsymbol{Q} = V$. Consider a function $f: \boldsymbol{Q} \rightarrow \boldsymbol{Q}$ which is defined as follows:

If $\frac{p}{q}$ is a rational number expressed in lowest terms (i.e., p & q have no prime factors in common), with q positive, then set $f\left(\frac{p}{q}\right) = \frac{1}{q}$. Determine whether f is continuous at any point?

Solution: As a matter of fact, f is discontinuous everywhere. Because the values of f are $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}..$ etc.

Now if we take a neighbourhood V of $\frac{1}{2}$ in the image set $\boldsymbol{Q}$, it will be typically an open set. We shall take this neighbourhood to be an interval, which would be small enough so that it would not contain any of the other image points say $\left(\frac{1}{1}, \frac{1}{3}, \frac{1}{4} \ldots\right)$ etc. Ultimately, we observe that:

$f^{-1}(V) = \left\{\cdots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\right\}$. In particular this one is not an open set. Hence f is not continuous.

6.4 How continuity helps evolving topology

The modern definitions of continuity (namely Definition 11a, 11b) pave topologist the new avenues to excavate the topological contents into deeper possible cores. And presently, topologists of the 21st century are leveraged with many advanced topological tools in terms of axioms, propositions, theorems and lemmas etc. Though, there are a variety of devices for topological continuity to attack on continuity problems, we would include here a few of them.

Theorem 1: Equivalent definitions of continuity(Munkres, 2000)

Let X & Y be topological spaces and let $f: X \rightarrow Y$ be a mapping. Then the following statements are equivalent:

- **a.** f is continuous
- **b.** For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$, where bar stands for closure.
- **c.** For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.
- **d.** For each $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. If this condition holds for the point $x \in X$, we say that f is continuous at the point x.

Examples based on Theorem 1:

First. A bijective function can be continuous- example as a witness is the identity function given by $g: \boldsymbol{R}_l \rightarrow \boldsymbol{R}$, where $\boldsymbol{R}_l$ stands for real set-in lower limit topology. This identity function is continuous because of the following arguments:
a. If we define the given identity function as- $g(x) = x$ for all real numbers x
b. Then the inverse image of open set (a, b) is equal to itself, which is open in $\boldsymbol{R}_l$.
c. However, in contrast to $g: \boldsymbol{R}_l \rightarrow \boldsymbol{R}$, if $h: \boldsymbol{R} \rightarrow \boldsymbol{R}_l$ be another identity function defines as $h(x) = x$ for all reals x. Then, h is not a continuous function. Because the inverse image of open set $[a, b)$ of $\boldsymbol{R}_l$ is equal to itself but it is not open in $\boldsymbol{R}$.

Second. Let S^1 stands for the unit circle defined by $S^1 = \{X \times Y \mid x^2 + y^2 = 1\}$. Further, consider a subspace of the plane $\boldsymbol{R}^2$, and let $F: [0,1] \rightarrow S^1$ be a map defined by $F(t) = (\cos 2\pi t, \sin 2\pi t)$. Then F is evidently continuous as the continuity follows due to familiar property of trigonometric functions.

As an extension of the Theorem 1, the following examples shall be more appropriate to held- that under continuous transformations, topological properties of the space remain preserved.

Example 6: A circle can be transformed into a square under a certain continuous transformation and the vice-versa is also possible (Figure 9).

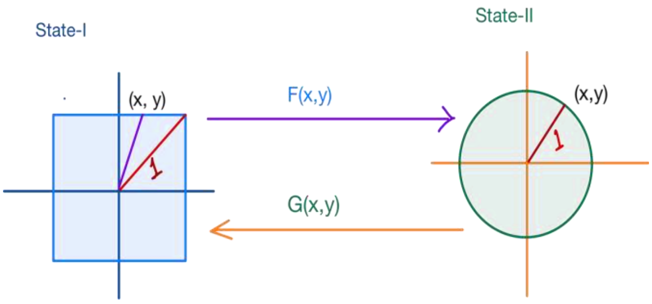

Figure 9: continuous transformation of a unit circle into a square of diagonal 2 and vice-versa

Example 7: As another example from topology, a coffee mug can be transformed into a doughnut and vice-versa under a continuous transformation (see S. M. Blinder "Coffee Mug to Donut" http://demonstrations.wolfram.com/CoffeeMugToDonut/ Published on March 7 2011)

Example 8: There is another popular notion of continuous function that sends an interval onto two disjoint intervals. That is, *any function that maps a single interval into two disjoint intervals exists only when function rips/ tears the domain of definition.* But ripping or tearing being prohibited specially in topology, therefore we can say that: *Topologically, it is almost impossible to develop a continuous transformation which could map an interval into two mutually disjoint intervals.* Thus, to make such a transformation continuous, one has to remove ripping or if ripping/ tearing is felt necessary in underlying transformation, it has to be repaired before completion of transformation action.

6.5 Why ripping or tearing is not a continuous function?

In Example 8, it is claimed that, it is nearly impossible to setup any continuous transformation, that could map an interval onto two disjoint intervals without ripping/ or tearing the domain interval. It does not mean that transformation involving tearing cannot made continuous, rather we can achieve continuity by diagnosing the ripping (if occurs) and then repairing it before the transformation completes its action, e.g., consider the following figure 11, wherein a knot (i) is mapped onto a circle (ii) by doing some kind of mathematical surgery for the injury/ cut on the knot before the transformation is completed.

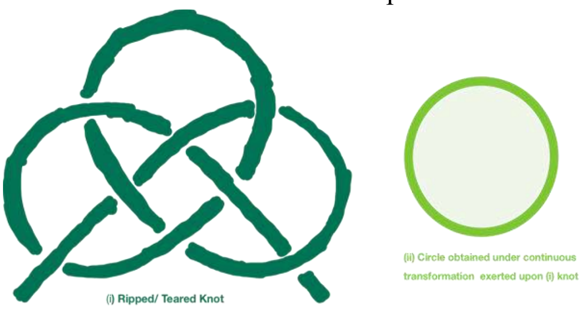

Figure 11: Knot (i) mapped onto circle (ii) under suitable continuous transformation with the aid of appropriate mathematical surgery done for the cut on the knot (i) before completion of the transformation.

Example 9: Transforming an entire real line into a single point is a kind of continuous transformation

Solution: Let us consider a point θ. Further let $Y = \{\theta\}$ be any non-empty set. Let us evolve a topological space (Y, τ) for Y such that the topology $\tau = \theta, \phi$. Then a map $\psi: R \to Y$, defined by $\psi(x) = \theta \; \forall \; x \in R$ is a continuous transformation of an entire real line to a point.

Further, the requirement of this example can be supplied through two ways- first by implementation of popular traditional criteria of continuity and second is by involving the modern topological continuity. Here, we try both of these approaches to obtain the solution.

First. **Traditional criteria of continuity-** We consider sets $\phi \neq X, Y \subset R$ such that $\alpha \in X$ and $\theta \in Y$. Further, assume a map $\psi: X \to Y$, defined by the rule $\psi(x) = \theta \; \forall \; x \; \in X$ i.e., the function defined in this fashion remains constant throughout its action on the domain of definition. Then, under this assumption, the function will map entire real line into a single point. For this, we assume that $\epsilon > 0$ is any real number. Let $\delta = 1$ (which is of arbitrary choice and we could have let it be ∞, if that were a number). Then, whenever the elements $x \in X$ be such that $0 < |x - \alpha| < \delta$, it is also the case that: $|\psi(x) - \psi(\alpha)| < \epsilon$ i.e., $|\psi(x) - \psi(\alpha)| = |\theta - \theta| = 0 < \epsilon$. Since ϵ was arbitrary, we have succeeded in finding $\delta > 0$ for every $\epsilon > 0$, which implies $Lim_{x \to \alpha} \psi(x) = \psi(\alpha)$ (by definition of limit), so ψ is continuous at α. This result implies that the transformation $\psi(x) =$ for all $x \in R$, i.e., $R \equiv (-\infty, +\infty)$can be mapped into a point $\theta \in Y$.

Second. **Modern criteria of continuity-** Suppose $\psi: X \to Y$ be a constant function, where X and Y be any two topological spaces, then to prove that ψ be continuous, we proceed like below:

Assume that ψ is defined as $\psi(x) = \theta \; \forall x \in X$. Taking any open set $G \backslash subseteq X$, then by definition of inverse image, $\psi^{-1}(O) = \{x \in X | \; \psi(x) \; \in \; O\} = \{x \; \in \; X | \theta \in O\}$, since $\psi(x) = \theta$ constant. So, if $\theta \; \in O$ then $\psi^{-1}(O) = X$ the whole space. But, if $\theta \; \notin O$, then $\psi^{-1}(O) = \phi$. Thus, in either case the inverse image $\psi^{-1}(O)$ is open in X, so by assumption that any arbitrarily chosen open set of Y has an inverse image in X, which is open too in X.

However, there is another approach to deal with example (9), which is popularly known as- the rule for constructing continuous function and is being discussed below(Munkres, 2000):

Theorem 2 'Constant function rule': If $\psi: X \to Y$ maps all of X into single point θ_0 of Y, then ψ is continuous.

If we implement this constant function rule to example (9), it naturally follows that- though the line and a single point are entirely different looking spaces, then also, a line (one dimensional entity) can be transformed continuously into a point (zero-dimensional entity), but the reverse process is impossible, i.e., we can not '*undo* 'this transformation.

Apart from the section 6, which is devoted to the topological continuity, there remains one more aspect of continuity to be tackled with closeness of metric. The following section briefly includes few popular results on metric continuity.

7. Continuity in terms of closeness of metric on real line

Hitherto, we have gone through many conditions which held that-

First. The continuity of any function under the characteristic signature of every continuity condition was to decide upon an explicit formulation of the statement "***a member $f(x)$ is close to the member $f(c)$, whenever the number x is close to c***".

Second. The advent of distance function for real numbers $\boldsymbol{R}$ has quantified the amount of the "***degree of nearness or closeness"*** of two numbers.

Third. The mathematical formulation of "***degree of nearness or closeness"*** in the sense of metric function led mathematician to ascribe continuity under the influence of metric functions.

Fourth. Primarily, an informal definition of continuous function aided with metric function was given as(Mendelson, 1990)- "*The function $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ is continuous at the number $c \in \boldsymbol{R}$, if given a prescribed degree of nearness, $f(x)$ must be within this prescribed degree of nearness to $f(a)$, whenever x is within some corresponding degree of nearness to c*"

Fifth. However, the fourth statement was formalized by notifying the symbols ϵ and δ for the phrases 'prescribed degree of nearness' and 'corresponding degree of nearness' respectively.

Sixth. Finally, a symbol $d = meric\ function$ was coined to correspond the phrase "degree of nearness".

Seventh. With these formulations, continuity in terms of metric function has got the form: **Definition 12:** Let (X, d_1) and (Y, d_2) be metric spaces, and let $f: D \subset X \rightarrow Y$ be a function. Then, this function is said to be continuous at some $x_0 \in D$ if given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that: $d_2\big(f(x), f(x_0)\big) < \epsilon$, whenever $d_1(x, x_0) < \delta\ \forall x \in X$.

Eighth. The idea of 'closeness 'can also be admitted when we think of continuity of sequences. In fact, the continuity in metric spaces can also be understood in the flavor of sequences. Here is a famous theorem of topology, which characterizes the continuity of any metric space in terms of the convergence of sequence in that space.:

Ninth. **Theorem 3:** Let (X, d_1) and (Y, d_2) be metric spaces, and let $f: D \subset X \rightarrow Y$ be a function. Then, this function is said to be continuous at some $x_0 \in D$ if and only if, whenever $x_n: n \in Z^+$ is a sequence in D that converges to x_0, then the sequence $f(x_n): n \in Z^+$ converges to $f(x_0)$ in Y

Tenth. In the light of above theorem, one can deduce that: *"Any function between any two metric spaces will be continuous, if and only if the function preserves the convergence of sequences."*

CONCLUSION

In the present digest, we have tried to encapsulate the notion of mathematical continuity from the perspective of ancient and modern masterpieces of wisdom. Specially, we have tried to include the arguments on continuum/ continua, infinitesimal and continuity proposed by four categories of wisdom seekers, namely Philosophers, Geometers, Arithmetizers and Topologists. We have taken Henri Poincaré's statement as an inspiration to this digest on continuity and initiated with the divisionism of Greek atomist philosopher Democritus (450 BC) and Eudoxus (350 BC). Afterwards, Aristotle's arguments on indivisibility and axioms of continuity have been discussed. Subsequently Simplicius fluxion, Euclidean postulates and elementary doctrine of continuity, and Killing's rule for intersection points have been studied. Further, in subsequent sections and subsections, 17th & 18th century polymaths like Leibnitz's monadism, Newton's fluxions calculus, Euler's refutation to monadism and Kantian phenomenal realm & noumenal realm are introduced. The approaches adapted by 19th and 20th century polymaths, namely closeness of Bolzano, limit perspective to continuity of Cauchy, epsilon-delta method of Weierstrass, Dedekind's postulates of continuity and Cantor's continuum with reference to 1-1 correspondence have been included in this digest. Finally, topological facets of continuity with special reference to Hausdorff's notion of neighborhood and a brief of 21st century continuity along with some examples have also been delineated. In a nutshell, in this digest, the notions of continuity, continua and infinitesimal have been revisited from almost each possible dimensions of the historical evolution of human intellect

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